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Positivstellensatz and flat functionals on path $*$-algebras

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ABSTRACT

We consider the class of non-commutative $*$-algebras which are path algebras of doubles of quivers with the natural involutions. We study the problem of extending positive truncated functionals on such $*$-algebras. An analog of the solution of the truncated Hamburger moment problem (Curto and Fialkow, 1991 [Fia91]) for path $*$-algebras is presented and non-commutative positivstellensatz is proved. We also present an analog of the flat extension theorem of Curto and Fialkow for this class of algebras.

1. Introduction

Let A be a $*$-algebra with a generating set $G = \{x_1, \dots, x_n, x_1^*, \dots, x_n^*\}$. Assume that there is a linear basis $\mathcal{B}$ consisting of words in the generators, such that $\mathcal{B} \cup \{0\}$ is multiplicatively closed. We will consider $\mathcal{B}$ as a linearly ordered set with respect to the degree-lexicographic order induced by a linear order on G. Then, a *moment sequence* $\alpha = (\alpha_b)_{b \in \mathcal{B}}$ $(\alpha_b \in \mathbb{C})$ gives rise to an infinite matrix M_α with rows and columns indexed by $\mathcal{B}$ such that $(M_\alpha)_{b_1, b_2} = \alpha_{b_1 b_2^*}$ $(b_1, b_2 \in \mathcal{B})$. The matrix M_α is called the *moment matrix* of α. The uppermost left $n(t) \times n(t)$-corner M_t of M, where $n(t)$ is the number of words of length less or equal to t is called the *truncated moment matrix* of α of order t. The matrix M_t depends only on the first $2t$ elements of the sequence α denoted by α_{2t} in sequel. The latter is called a truncated moment sequence. Thus we can write $M_t = M(\alpha_{2t})$. We will denote by A_t the subspace of A spanned by the words b of length less or equal to t.

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When A is the polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$ and $x_j^* = x_j$ the classical (full) moment problem asks for which $\alpha = (\alpha_b)_{b \in \mathcal{B}}$ there is a representing measure μ, i.e. a positive Borel measure on $\mathbb{R}^n$ such that for each monomial $b = b(x_1, \dots, x_n)$:

$$\alpha_b = \int\limits_{\mathbb{R}^n} b(x_1, \dots, x_n)\, d\mu(x_1, \dots, x_n).$$

The question if there is such a measure μ with the support contained in a set $K \subseteq \mathbb{R}^n$ is called the K-moment problem. A solution of the K-moment problem is given by the classical Riesz–Haviland theorem. However, the natural analogs of the moment problem and the K-moment problem for truncated moment sequences are more delicate [Fia96,Fia05,Fia08]. We mention the general solution of the truncated K-moment problem in terms of extensions (called the truncated version of the Riesz–Haviland theorem in [Fia91]) and the solution in case of a flat α (see [Fia96]) because of their relevance to the present paper.

Theorem. *(See Curto and Fialkow [Fia08].) A truncated moment sequence α_t has a K-representing measure if and only if α_t admits a K-positive extension α_{t+1}.*

Here, K-positivity means that the *Riesz functional* $L_{\alpha_t} : \mathbb{R}[x_1, \dots, x_n]_t \to \mathbb{R}$ defined as $L_{\alpha_t}(\sum_b \xi_b b) = \sum_b \xi_b \alpha_b$ attains positive values on $p = \sum_{b \in \mathcal{B}} \xi_b b \in \mathbb{R}[x_1, \dots, x_n]_t$ ($\xi_b \in \mathbb{R}$) such that $p(K) \subseteq [0, +\infty)$.

For the non-commutative $*$-algebras A the natural analog of a measure is a representation on a Hilbert space. We will say that α (or M) is *representable* if there is a representation $\pi : A \to L(H)$ on a pre-Hilbert space H and $\xi \in H$ such that

$$\alpha_b = \langle \pi(b)\xi, \xi \rangle.$$

If J is an ideal in A and $\pi(J) = 0$ then we will say that α is *A/J-representable*. It is easy to see that for the non-commutative *free* $*$-algebra $\mathcal{F}$ the full moment problem is more simple than in the polynomial algebra case. Namely, the full moment problem is solvable if and only if M is positive semidefinite. An answer to the truncated moment problem for the free $*$-algebras can be given in terms of extensibility (see [Put04]). More precisely, a sequence α_t is representable if and only if it admits an extension α_{t+1} such that M_{t+1} is positive semidefinite. Moreover, if this is the case, there is a representation π acting on a finite-dimensional Hilbert space and a vector ξ such that $\alpha_b = \langle \pi(b)\xi, \xi \rangle$ for $|b| \leqslant t$. The same holds for $\mathcal{F}/J$-moment problem for some non-trivial ideals J, for example, for the ideals defining spherical isometries (see [Put04]). The complete analog of this statement is not true for arbitrary ideals. For instance, there are ideals $J \subseteq \mathcal{F}$ such that all non-trivial representations of $\mathcal{F}/J$ necessarily have unbounded operators in their images. For any such representation π, the sequence $\alpha_t = (\langle \pi(b)\xi, \xi \rangle)_{|b| \leqslant t}$, clearly, has a positive extension, but there is no representation τ in *bounded* operators which define α_t with $\tau(J) = \{0\}$. To see other possible pitfalls, consider the $*$-algebra $A = \mathbb{C}\langle x, x^* \mid (xx^*)^n = 0 \rangle$. If π is a representation of the free algebra $\mathcal{F}$, and $2t < n$ then the matrix M_t corresponding to $\alpha_t = (\langle \pi(b)\xi, \xi \rangle)_{|b| \leqslant t}$ in the algebra $\mathcal{F}$ is the same as the matrix M_t corresponding to α_t in A. Hence, there is a positive definite matrix M_t for every t. Since A has no non-trivial representations, the truncated sequence α_t is never A-representable.

Some results on the decompositions of non-commutative polynomials which are positive in every representation into sums of hermitian squares in the free $*$-algebras were obtained in [Hel02], and, for some other classes of algebras, in [Put04,Sch08]. In the present paper we study the non-commutative moment problem for the path $*$-algebras $\mathcal{A}_\Gamma$ associated with the finite graphs Γ. Following ideas of [Put04], we show in Section 3 that a moment sequence α_t is representable if and only if there is a positive semidefinite extension α_{t+1}. Moreover, we show that every hermitian element f which is mapped to a positive semidefinite operator by every finite-dimensional representation of $\mathcal{A}_\Gamma$ is a sum of hermitian squares. Together these facts may be seen as an abstract solution to the non-commutative (full) moment problem for $\mathcal{A}_\Gamma$.

The other case when the K-moment problem for the polynomial algebra has an especially tractable solution is the case of a *flat* data α_t. The latter means that the rank of $M(\alpha_t)$ is equal to the rank of $M(\alpha_{t-1})$. In this case, α_t admits a representing measure if and only if M_t is positive semidefinite [Fia96]. Moreover, there is an r-atomic representing measure with $r = \operatorname{rank} M(\alpha_t)$. This is a consequence of the following “flat extension theorem”.

Theorem. *(See Curto and Fialkow [Fia96].) Every flat L_{α_t} on $\mathbb{R}[x_1, \dots, x_n]_{2t}$ can be extended to a flat L on $\mathbb{R}[x_1, \dots, x_n]_{2t+2}$.*

In [Lau05,Lau08] M. Laurent has given an algebraic approach to the Curto and Fialkow flat extension theorem. The Gröbner bases theory for the polynomial algebras is used in several technical constructions in these papers. In Section 4 we translate ideas of M. Schweighofer's unpublished manuscript on Gröbner bases approach to the Curto and Fialkow flat extension theorem into the non-commutative setting. We prove the following analog of the Curto and Fialkow flat extension theorem: every flat truncated functional L_t on $\mathcal{A}_\Gamma$ admits a flat extension L_{t+1}. In particular, every positive flat truncated functional admits an extension to a positive flat functional on $\mathcal{A}_\Gamma$ and, thus, via GNS construction, defines a finite-dimensional representation of $\mathcal{A}_\Gamma$. It is interesting in view of the Lance–Tapper conjecture (see Section 2 and [Lan97,Tap99]), to obtain an analog of the extension theorem for the flat functionals supported on $\mathcal{A}_\Gamma/J$, where J is an ideal generated by paths (i.e. $\mathcal{A}_\Gamma/J$ is a monomial $*$-algebra). In the last section we give some conditions which ensure that a truncated functional on the free $*$-algebra has a flat extension.

In a forthcoming paper we generalize the extension theorem for a class of $*$-algebras containing all monomial $*$-algebras. In the present paper we restrict ourselves to the class of path $*$-algebras to minimize the use of Gröbner basis theory and keep formulations of the main results close to the classical (commutative) analogues. The reason of this simplification is the fact that Gröbner basis theory for the right ideals is much simpler for the path $*$-algebras than for the monomial algebras and the commutative polynomial algebras (see [Gre00]).

2. Definitions

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a *quiver*, that is a directed graph with a finite set of vertices Γ_0 and a finite set of arrows Γ_1. Every arrow b has the unique origin vertex $o(b)$ and the terminal vertex $t(b)$. For vertices $e_1, e_2 \in \Gamma_0$ we denote by $\Gamma(e_1, e_2)$ the set of arrows b with $o(b) = e_1$, $t(b) = e_2$. A path p in Γ is a finite sequence of arrows (possibly empty) $(v_1, \dots, v_k)$ such that $o(b_{j+1}) = t(b_j)$ for $j = 1, \dots, k-1$. The number k is called the *length* of p. The unique empty path at vertex e will be denoted also by e.

Consider *the double* Γ^* of the quiver Γ which is the quiver with the same set of vertices $\Gamma_0^* = \Gamma_0$ and doubled number of arrows $(\Gamma^*)_1 = \Gamma_1 \cup \Gamma_1^*$, where $\Gamma_1^* = \{b^* \mid b \in \Gamma_1\}$ and if $b \in \Gamma(e_1, e_2)$ then $b^* \in \Gamma^*(e_2, e_1)$. Clearly, the double Γ^* does not depend on the orientation of the arrows of Γ and, thus, is defined by the underlying graph of the quiver Γ.

The set $\mathcal{B}$ consisting of the paths in a quiver Γ together with the zero element 0 is a semigroup with multiplication given by concatenation of paths (the product $b_1 b_2$ is 0 if $t(b_1) \neq o(b_2)$). In particular, for every vertex $e \in \Gamma_0$ this semigroup contains an idempotent (denoted also by e) corresponding to the trivial path at vertex e. The semigroup algebra $\mathbb{C}\Gamma$ is called *the path algebra* of Γ. Clearly, $\mathcal{B}$ is a linear basis of $\mathbb{C}\Gamma$. The path algebra of Γ^* is a $*$-algebra with the involution which maps b to b^* ($b \in \mathcal{B}$). We will denote this $*$-algebra by $\mathcal{A}_\Gamma$ in sequel.

In particular, for a quiver Γ with one vertex and n arrows $x_1, \dots, x_n$, the algebra $\mathcal{A}_\Gamma$ is the free $*$-algebra $\mathbb{C}\langle x_1, \dots, x_n, x_1^*, \dots, x_n^*\rangle$. Consider an arbitrary order on the vertices $v_1 < \dots < v_r$ and arrows $v_r < a_1 < a_2 < \cdots$ such that every arrow is greater than every vertex. Having an order on $\Gamma_0 \cup \Gamma_1$ we can equip the set of paths $\mathcal{B}$ with the left degree-lexicographic order. Consider the $*$-algebra

$$A_{w_1,\dots,w_m} = \mathbb{C}\big\langle x_1, \dots, x_n, x_1^*, \dots, x_n^* \bigm| w_1 = 0, \dots, w_m = 0, w_1^* = 0, \dots, w_m^* = 0\big\rangle,$$

where w_j are words in $x_1, \ldots, x_m, x_1^*, \ldots, x_m^*$. We can always assume that the set $\{w_1, \ldots, w_m, w_1^*, \ldots, w_m^*\}$ is reduced, i.e. no word from this set is a subword of some other word from this set. Such algebras $A_{w_1,\ldots,w_m}$ constitute a subclass of the *monomial* algebras. A word w is called *unshrinkable* if w cannot be presented as udd^* or dd^*u for some word u and non-empty word d. We call an algebra $A_{w_1,\ldots,w_m}$ a Lance–Tapper $*$-algebra if w_j is unshrinkable for $j = 1, \ldots, m$. It was conjectured by Lance and Tapper that a $*$-algebra $A_{w_1,\ldots,w_m}$ has a faithful representation on a Hilbert space if and only if it is a Lance–Tapper $*$-algebra (see [Lan97,Tap99,Pop02]). Moreover, they conjectured that every Lance–Tapper $*$-algebra has a separating family of finite-dimensional representations. The following simple lemma shows that conjecture is true for the trivial monomial ideals.

Lemma 1. *For any quiver Γ the path $*$-algebra $\mathcal{A}_\Gamma$ has a separating family of finite-dimensional representations.*

Proof. For the path $*$-algebra $\mathcal{A}_\Gamma$ there is a $*$-isomorphism ϕ of $\mathcal{A}_\Gamma$ onto a subalgebra in $M_n(\mathcal{F}) = \mathcal{F} \otimes M_n$, where $n = |\Gamma_0|$ and $\mathcal{F}$ is the free $*$-algebra with the free generators x_b corresponding to each $b \in \Gamma_1$. Let $e_1, \ldots, e_n$ be an enumeration of the vertices of Γ. Under the isomorphism ϕ, any arrow $b \in \Gamma(e_i, e_j)$ is sent to $x_b \otimes E_{ij}$, and the idempotents e_i are sent to E_{ii}. Since the free $*$-algebra $\mathcal{F}$ has a separating family of finite-dimensional representation (see [Avi82,Pop02]) the same is true for any path $*$-algebra. $\square$

There is a well-known bijective correspondence between the representations of $\mathcal{A}_\Gamma$ and the representations of Γ in the category of Hilbert spaces. Recall that a representation Π of Γ in the category of Hilbert spaces is a mapping which sends each vertex $e \in \Gamma_0$ into a Hilbert space H_e, and each arrow $b \in \Gamma(e_1, e_2)$ into a linear operator $\Pi(b) : H_{e_1} \to H_{e_2}$. To a representation Π of Γ there corresponds a representation π of $\mathcal{A}_\Gamma$ in the Hilbert space $H = \bigoplus_{i=1}^n H_{e_i}$ such that, for every $b \in \Gamma(e_i, e_j)$, the operator $\pi(b) : H \to H$ is given by the block-matrix $\Pi(b) \otimes E_{ij}$, and $\pi(b^*)$ is defined as the adjoint operator to $\pi(b)$. Conversely, given a representation π of $\mathcal{A}_\Gamma$ in a Hilbert space, we put $H_e = \pi(e)(H)$ (here e is the trivial path at the vertex e). Since for $b \in \Gamma(e_1, e_2)$ we have $e_1 b e_2 = b$, one can easily check, that the operator $\pi(b)$ maps the subspace H_{e_2} into the subspace H_{e_1}. Thus, the mapping $e \mapsto H_e$, $b \mapsto \pi(b)|_{H_{e_2}}$ defines a representation of Γ.

3. Sums of squares decomposition

Enumerating the arrows Γ_1 as $b_1, \ldots, b_n$, we get a one-to-one correspondence between the class of finite-dimensional representations $\mathrm{Rep}_{f.d.}\,\mathcal{A}_\Gamma$ of $\mathcal{A}_\Gamma$ and the class Z of n-tuples of operators $(X_1, \ldots, X_n)$ acting on the finite-dimensional Hilbert spaces such that

$$X_i X_j = 0 \quad \text{if } b_i b_j = 0, \tag{1}$$

$$X_i^2 = X_i \quad \text{if } b_i = b_i^2, \tag{2}$$

$$X_i X_j = X_i \quad \text{if } b_i b_j = b_i, \tag{3}$$

$$X_i X_j = X_j \quad \text{if } b_i b_j = b_j. \tag{4}$$

Clearly, $\mathcal{A}_\Gamma$ is isomorphic to the quotient of the free $*$-algebra $\mathcal{F}$ with the generating set $X = \Gamma_1 \cup \Gamma_1^*$ modulo the $*$-ideal generated by relations (1)–(4). Clearly, for any unitary operator $U : H \to K$ between two Hilbert spaces, n-tuple $(UA_1U^*, \ldots, UA_nU^*)$ belongs to Z. Hence, Z is closed under the joint unitary transformations. The following lemma is a direct consequence of [Put04] and Lemma 1. Recall that $(\mathcal{A}_\Gamma)_d$ denotes the subspace generated by the paths of length no greater than d.

Lemma 2. *For any $d \geqslant 1$ the cone $\mathcal{C}_{2d}(\mathcal{A}_\Gamma) = \mathrm{co}\{ff^* \mid f \in (\mathcal{A}_\Gamma)_d\}$ is closed in $(\mathcal{A}_\Gamma)_{2d}$. Here,* co *denotes convex hull.*

Proof. The ideal $I(Z)$ in the free algebra $\mathcal{F}$ consisting of all elements $p(x_1,\dots,x_n,x_1^*,\dots,x_n^*)$ such that $p(X_1,\dots,X_n,X_1^*,\dots,X_n^*)=0$ for all $(X_1,\dots,X_n)\in Z$ coincides with the kernel of the canonical surjection $\psi:\mathcal{F}\to\mathcal{A}_\Gamma$ by Lemma 1. The mapping ψ maps $\operatorname{co}\{ff^*\mid f\in\mathcal{F}_d\}$ to a closed subset in $\mathcal{P}_{2d}=\psi(\mathcal{F}_{2d})$ by [Put04, Lemma 3.2]. Let us denote this subset by $\mathcal{C}_{2d}(Z)$. Identifying $\mathcal{F}/I(Z)$ with $\mathcal{A}_\Gamma$ via ψ, we have that $\mathcal{P}_{2d}=(\mathcal{A}_\Gamma)_{2d}$, and $\mathcal{C}_{2d}(Z)=\mathcal{C}_{2d}(\mathcal{A}_\Gamma)$. Hence, the lemma follows. $\square$

Lemma 3. *Let H be a pre-Hilbert space, $d\geqslant 1$ and $H_0\subseteq H_1\subseteq\cdots\subseteq H_d=H$ be subspaces with H_0 finite-dimensional. Assume that for $0\leqslant t\leqslant d-1$, we are given a linear map $\pi_t:(\mathcal{A}_\Gamma)_t\to L(H_{d-t},H)$ such that*

(1) *for every $f\in(\mathcal{A}_\Gamma)_t$ and $0\leqslant s\leqslant d-t$, $\pi_t(f)(H_{d-t-s})\subseteq H_{d-s}$;*
(2) *for every $0\leqslant r\leqslant t$ and $g\in(\mathcal{A}_\Gamma)_{t-r}$ we have $\pi_{t-r}(g)|_{H_{d-t}}=\pi_t(g)$. Thus, we can omit the subscript t in the notation $\pi_t(f)$;*
(3) *for every $f_1\in(\mathcal{A}_\Gamma)_{t_1}$, $f_2\in(\mathcal{A}_\Gamma)_{t_2}$ with $t=t_1+t_2\leqslant d-1$, we have $\pi(f_1f_2)|_{H_{d-t}}=\pi(f_1)\pi(f_2)|_{H_{d-t}}$;*
(4) *for $f\in(\mathcal{A}_\Gamma)_t$ and $u,v\in H_{d-t}$,*

$$\langle\pi(f)u,v\rangle=\langle u,\pi(f^*)v\rangle.$$

Then, there is a finite-dimensional subspace $H'\subseteq H$ with $\dim H'\leqslant\dim H_0\dim(\mathcal{A}_\Gamma)_d$, $H_0\subseteq H'$ and a representation τ of $\mathcal{A}_\Gamma$ on H' such that

$$\pi(a)|_{H_0}=\tau(a)|_{H_0}\quad\textit{for all } a\in(\mathcal{A}_\Gamma)_{d-1}.$$

Proof. Put $H'=\{\pi(f)H_0\mid f\in(\mathcal{A}_\Gamma)_d\}$, $K=\{\pi(f)H_0\mid f\in(\mathcal{A}_\Gamma)_{d-1}\}$, and $V=\{\pi(f)H_0\mid f\in(\mathcal{A}_\Gamma)_{d-2}\}$ which are finite-dimensional subspaces of H. We will define a representation Π of Γ. For $e\in\Gamma_0$, $\pi(e)$ is a projection defined on H. Put $H_e=\pi(e)H'$ and $K_e=\pi(e)K$. Let $K_e^\perp$ be the orthogonal complement of K_e in H_e, i.e. $H_e=K_e\oplus K_e^\perp$.

For $b\in\Gamma_1(e_1,e_2)$, we put $\Pi(b)|_{K_{e_2}}=\pi(b)|_{K_{e_2}}$ and $\Pi(b)|_{K_{e_2}^\perp}=0$. Since $\pi(e_1)\pi(b)=\pi(b)$ on H_{d-1}, we have $\Pi(b):H_{e_2}\to H_{e_1}$ and $K_{e_2}^\perp\subseteq\ker\Pi(b)$. Hence, for the adjoint operator $\Pi(b)^*$, we have

$$\Pi(b)^*:H_{e_1}\to H_{e_2}\quad\text{and}\quad\operatorname{Ran}\Pi(b)^*\subseteq K_{e_2}.$$

From this follows that

$$\pi\left(b^*\right)\big|_{K_{e_1}\cap V}=\Pi(b)^*\big|_{K_{e_1}\cap V}. \tag{5}$$

Indeed, for any $v\in K_{e_1}\cap V$ and $w\in K_{e_2}$, we have

$$\langle w,\Pi(b)^*v\rangle=\langle\Pi(b)w,v\rangle=\langle\pi(b)w,v\rangle=\langle w,\pi\left(b^*\right)v\rangle.$$

Since $\Pi(b)^*v\in K_{e_2}$, $\pi(b^*)v\in K_{e_2}$ and $w\in K_{e_2}$ is arbitrary, we get $\Pi(b)^*v=\pi(b^*)v$. Let τ be the representation of $\mathcal{A}_\Gamma$ corresponding to Π. Then, τ is defined on the Hilbert space $\bigoplus_e H_e$ which we will identify with H'. By (5) and induction in length of path $b\in\mathcal{B}$, we get that $\tau(b)|_{H_0}=\pi(b)|_{H_0}$ for all paths b with $|b|\leqslant d-1$. $\square$

Remark 4. If π is a (possibly unbounded) representation of $\mathcal{A}_\Gamma$ on a pre-Hilbert space H and $H_0\subseteq H$ is a finite-dimensional subspace, then with $H_t=\pi((\mathcal{A}_\Gamma)_t)H_0$, we have, for every $f\in(\mathcal{A}_\Gamma)_t$, the inclusion $\pi(f)(H_{d-t})\subseteq H_d$ and conditions (1)–(4) are satisfied. Hence, there is a finite-dimensional representation τ such that $\pi(f)|_{H_0}=\tau(f)|_{H_0}$ for $f\in(\mathcal{A}_\Gamma)_{d-1}$.

The following corollary is an analog of the solution of the truncated Hamburger moment problem from [Fia91].

Corollary 5. *A truncated linear functional L_d on a path $*$-algebra $(\mathcal{A}_\Gamma)_{2d}$ admits an extension to a positive functional L on $\mathcal{A}_\Gamma$ if and only if it admits and extension to a positive functional $L_{d+1} : (\mathcal{A}_\Gamma)_{2d+2} \to \mathbb{C}$.*

Proof. Let $K = \{f \in (\mathcal{A}_\Gamma)_{2d+2} \mid L_{d+1}(ff^*) = 0\}$. Since L_{d+1} is positive,

$$\left|L_{d+1}\left(fg^*\right)\right| \leqslant L_{d+1}\left(ff^*\right)^{1/2} L_{d+1}\left(gg^*\right)^{1/2}$$

and K is the null space of the sesquilinear form $\langle f, g\rangle = L_{d+1}(fg^*)$ defined on the space $(\mathcal{A}_\Gamma)_{2d+2}$. Then, $\langle\cdot,\cdot\rangle$ induces an inner product on the quotient space $H = (\mathcal{A}_\Gamma)_{d+1}/K$. Let $H_t = (\mathcal{A}_\Gamma)_t + K \subseteq H$ and let H_0 be the 1-dimensional subspace generated by $\xi = 1 + K$. Given $f \in (\mathcal{A}_\Gamma)_t$, define $\pi_t(f)$ as the restriction on H_{d+1-t} of the operator of right multiplication by f. It is routine to check conditions (1)–(4) of Lemma 3. Thus, there is a finite-dimensional representation τ such that, for all $f, g \in (\mathcal{A}_\Gamma)_d$, we have

$$L_d\left(fg^*\right) = \left\langle \pi_d(f)\xi, \pi_d(g)\xi\right\rangle = \left\langle \tau(f)(\xi), \tau(g)\xi\right\rangle = \left\langle \tau\left(fg^*\right)(\xi), \xi\right\rangle.$$

This proves that L has the positive extension $\langle \tau(\cdot)(\xi), \xi\rangle$. □

Let $\mathrm{Sym}_d(\Gamma)$ be the set consisting of the hermitian elements f of $\mathcal{A}_\Gamma$ with $\deg(f) \leqslant d$. By a straightforward reformulation of [Put04, Lemma 3.1] combined with Lemma 3, we get the following lemma.

Lemma 6. *For every $d \geqslant 0$ there is a basis $\beta_1, \ldots, \beta_k$ of $\mathrm{Sym}_d(\Gamma)$, a sequence $\pi_1, \ldots, \pi_k$ of representations of $\mathcal{A}_\Gamma$ on a finite-dimensional Hilbert space H and a vector $\xi \in H$ such that for all i, j:*

$$\left\langle \pi_i(\beta_j)\xi, \xi\right\rangle = \delta_{ij}, \tag{6}$$

where δ_{ij} is Kronecker's symbol.

Following [Put04], we introduce a norm $\|\cdot\|$ on $\mathrm{Sym}_d(\Gamma)$ by the formula

$$\|f\| = \sum_{i=1}^{k} \left|\left\langle \pi_i(f)\xi, \xi\right\rangle\right|, \tag{7}$$

for $f \in \mathrm{Sym}_d(\Gamma)$. For even d and any $h \in \mathcal{C}_d(\mathcal{A}_\Gamma)$, $\|h\|$ can be expressed as a value of the linear functional $N(h) = \sum_{i=1}^{k} \langle \pi_i(h)\xi, \xi\rangle$, i.e. $\|h\| = N(h)$.

The following theorem shows that a hermitian element of $\mathcal{A}_\Gamma$ which is positive semidefinite in every finite-dimensional representation is a sum of hermitian squares.

Theorem 7. *If a hermitian $q \in (\mathcal{A}_\Gamma)_{d-1}$ is such that for any representation π of $\mathcal{A}_\Gamma$ with $\dim \pi \leqslant \dim(\mathcal{A}_\Gamma)_d$ we have that $\pi(q)$ is positive semidefinite then $q \in \mathcal{C}_{2d}(\mathcal{A}_\Gamma)$.*

Proof. Assume that $q \notin \mathcal{C}_{2d}(\mathcal{A}_\Gamma)$. Since $\mathcal{C}_{2d}(\mathcal{A}_\Gamma)$ is closed in $\mathrm{Sym}_d(\Gamma)$, Minkowski's separation theorem implies that there is a linear functional $L_0 : (\mathcal{A}_\Gamma)_{2d} \to \mathbb{C}$ such that $L_0(q) < 0 \leqslant \min_{c \in \mathcal{C}_{2d}} L_0(c)$. Take $\epsilon > 0$ then, for $L = L_0 + \epsilon N$, we have $L(ff^*) > 0$ for $f \in (\mathcal{A}_\Gamma)_d \setminus \{0\}$. Hence, $\langle f, g\rangle = L(fg^*)$ defines a scalar product on $H = (\mathcal{A}_\Gamma)_d$. By choosing ϵ we can have $L(q) < 0$. Let $H_0 = \mathbb{C}1$ and $H_t = (\mathcal{A}_\Gamma)_t$ $(t \leqslant d)$. For $g \in (\mathcal{A}_\Gamma)_t$ let $R_g : H_{d-t} \to H_d$ denote the operator of multiplication by g from the right. By Lemma 3 there is a finite-dimensional representation τ in a Hilbert space H' such that $H_0 \subseteq H'$ and $\dim H' \leqslant \dim(\mathcal{A}_\Gamma)_d$ and such that $\tau(g)1 = R_g 1 = g$ for all $g \in (\mathcal{A}_\Gamma)_{d-1}$. Hence, $L(q) = \langle R_q 1, 1\rangle = \langle \tau(q)1, 1\rangle \geqslant 0$. This contradiction proves that $q \in \mathcal{C}_{2d}(\mathcal{A}_\Gamma)$. □

4. Flat functionals

In this section we study the question which truncated positive semidefinite functional on a path $*$-algebra can be extended to a positive semidefinite functional on the whole algebra with the moment matrix of finite rank. We will need some standard definitions from Gröbner basis theory. In what follows, all associative algebras will be considered over the field of complex numbers. For an algebra A to have Gröbner basis theory, A must have a multiplicative linear basis $\mathcal{B}$ (i.e. for every $b_1, b_2 \in \mathcal{B}$, $b_1 b_2 \in \mathcal{B}$ or $b_1 b_2 = 0$) with an admissible order on $\mathcal{B}$ (see [Gre00]). An order $>$ is called *admissible* if

A0. $>$ is well-order on $\mathcal{B}$.
A1. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_1 b_3 > b_2 b_3$ if both $b_1 b_3$ and $b_2 b_3$ are non-zero.
A2. For all $b_1, b_2, b_3 \in \mathcal{B}$, if $b_1 > b_2$ then $b_3 b_1 > b_3 b_2$ if both $b_3 b_1$ and $b_3 b_2$ are non-zero.
A3. For all $b_1, b_2, b_3, b_4 \in \mathcal{B}$, if $b_1 = b_2 b_3 b_4$ then $b_1 \geqslant b_3$.

Following [Gre00], we say that A has an ordered multiplicative basis $(\mathcal{B}, >)$ if $\mathcal{B}$ is a multiplicative basis and $>$ is an admissible order on $\mathcal{B}$. It was shown in [Gre00], that every algebra with an ordered multiplicative basis is a quotient of a path algebra modulo some 2-nomial ideal. By a 2-*nomial* ideal we mean an ideal generated by some elements of the form p or $p - q$, where $p, q \in \mathcal{B}$. An algebra A which is a quotient of a path algebra modulo an ideal generated by a set of paths is called *monomial*.

Let $x = \sum_{i=1}^{r} \alpha_i b_i$, where $\alpha_i \in \mathbb{C}^*$, and b_i are distinct elements of $\mathcal{B}$. The tip of x, denoted by $\mathrm{Tip}(x)$, is the largest element in $\{b_1, \dots, b_r\}$, i.e. $\mathrm{Tip}(x) = b_j$ for b_j such that $b_j \geqslant b_i$ for all $i = 1, \dots, r$.

For $k \geqslant 1$, put $S_k = \{a \in \mathcal{B} \mid |a| \leqslant k\}$. Here, $|a|$ is the length of a, i.e. the minimum t such that a can be written as a product $b_1 \dots b_t$ with some $b_1, \dots, b_t \in \mathcal{B}$. Denote by V_k the linear span of S_k, and by V_k^* the dual vector space of V_k. Note that V_k denotes the same object as A_k in the preceding sections. We change the notation here to avoid the ambiguous notation A_k^*.

Definition 8. Given $L_k \in V_{2k}^*$, define $B_{L_k} : V_k \times V_k \to \mathbb{C}$ by the rule $B_{L_k}(p, q) = L_k(pq^*)$. A functional L_k with $k > 1$ (and a sesquilinear form B_{L_k}) will be called a flat truncated functional on A (resp. a flat sesquilinear form on A) if L_k is hermitian (i.e. $L_k(a^*) = \overline{L_k(a)}$ for all $a \in A$) and $\operatorname{rank} B_{L_k} = \operatorname{rank} B_{L_{k-1}}$, where L_{k-1} is the restriction of L_k to V_{k-1}.

Note, that the sesquilinear form B_{L_k} is given by the moment matrix M_α (where $\alpha_b = L_k(b)$) in the basis S_k of the vector space V_k.

Clearly, $\ker B_{L_k} \cap V_{k-1} \subseteq \ker B_{L_{k-1}}$, and we have a surjective linear maps π and an injective linear map i such that

$$V_{k-1}/\ker B_{L_{k-1}} \xleftarrow{\ \pi\ } V_{k-1}/(\ker B_{L_k} \cap V_{k-1}) \xhookrightarrow{\ i\ } V_k/\ker B_{L_k}. \tag{8}$$

In the above diagram, the dimensions are weakly increasing from left to right. Thus, if L_k is flat then π and i are isomorphisms. Hence, L_k is flat if and only if the following two conditions hold:

$$\ker B_{L_k} \cap V_{k-1} = \ker B_{L_{k-1}}, \tag{9}$$

$$V_k = V_{k-1} + \ker B_{L_k}. \tag{10}$$

We have the following analog of the *recursively generated* property of $\ker B_{L_k}$ which was studied for the commutative polynomial algebras in [Fia08].

Lemma 9. *Let $A = \mathcal{A}_\Gamma$ be a path $*$-algebra. If L_k is flat then, for every $p \in \ker B_{L_k}$ and $w \in \mathcal{B}$ such that* $\mathrm{Tip}(p)w \neq 0$ *and $pw \in V_k$, we have $pw \in \ker B_{L_k}$.*

Proof. We can assume that w belongs to the set S_1 (otherwise, decompose $w = w_1 \dots w_s$ and use induction in s). Given $v \in V_k$, we can decompose $v = h + g$, where $h \in V_{k-1}$ and $g \in \ker B_{L_k}$. Hence,

$$L_k\big(pwv^*\big) = L_k\big(p\big(wh^*\big)\big) + \overline{L_k\big(g(pw)^*\big)} = L_k\big(p\big(wh^*\big)\big) = 0.$$

The second equality follows from the inclusions $g \in \ker B_{L_k}$ and $pw \in V_k$. The last one follows from the inclusions $p \in \ker B_{L_k}$ and $wh^* \in V_k$. $\square$

We will show that any flat truncated functional L_k on a path $*$-algebra A can be extended to a hermitian functional on A of the same rank. For this we need basic facts about right Gröbner bases for right ideals. There is a general theory of right Gröbner bases for the right modules with coherent bases over the algebras with ordered multiplicative bases [Gre00]. Recall, that a linear basis $\mathcal{M}$ of a right module M over an algebra A with an ordered multiplicative basis $(\mathcal{B}, >)$ is called *coherent* if for all $m \in \mathcal{M}$ and every $b \in \mathcal{B}$ either $mb \in \mathcal{M}$ or $mb = 0$. A well-order $\succ$ on $\mathcal{M}$ is a *right admissible order on* $\mathcal{M}$ if

(1) for all $m_1, m_2 \in \mathcal{M}$ and $b \in \mathcal{B}$, if $m_1 \succ m_2$ then $m_1 b \succ m_2 b$ if both $m_1 b$ and $m_2 b$ are non-zero,
(2) for all $m \in \mathcal{M}$ and $b_1, b_2 \in \mathcal{B}$, if $b_1 > b_2$ then $m_1 b \succ m_2 b$ if both mb_1 and mb_2 are non-zero.

If $x \in M \setminus \{0\}$ then $x = \sum_{j=1}^{s} \alpha_j m_j$, where $\alpha_j \in \mathbb{C}^*$ and $m_1, \dots, m_s$ are distinct elements of $\mathcal{M}$. The tip of x, denoted by $\mathrm{Tip}(x)$, is the element m_i such that $m_j \not\succ m_i$ for all $i = 1, \dots, s$.

We will recall here the algorithm (see [Gre00]) of constructing a right Gröbner basis for a submodule M with a coherent ordered basis $(\mathcal{M}, \succ)$ of a right projective module over a path $*$-algebra A with an ordered multiplicative basis $(\mathcal{B}, >)$. This algorithm will be needed in the proof of Theorem 10. Assume that M is generated by a finite set of elements $H = \{h_1, \dots, h_n\}$. For a subset $X \subseteq M$ denote $\mathrm{Tip}(X) = \{\mathrm{Tip}(x) \mid x \in X\}$ and $\mathrm{NonTip}(X) = \mathcal{M} \setminus \mathrm{Tip}(X)$. If $\mathcal{N}$ is a right submodule of M then a subset $\mathcal{G} \subset \mathcal{N}$ is called a *right Gröbner basis* of $\mathcal{N}$ if $\mathrm{Tip}(\mathcal{G})$ generates $\mathcal{N}$ as a submodule. Consider the following algorithm of transforming the subset H.

(1) Remove 0 from H.
(2) Put $\mathcal{T}_H = \{\mathrm{Tip}(h) \mid \text{for all } h' \in H \setminus \{h\},\ \mathrm{Tip}(h') \text{ does not left divide } \mathrm{Tip}(h)\}$.
(3) For every $t \in \mathcal{T}_H$, choose $h \in H$ such that $\mathrm{Tip}(h) = t$ and renumber so that these elements are $h_1, \dots, h_s$. If $s = n$, we are done. Otherwise, denote by $Q^\dagger$ the right submodule generated by $h_1, \dots, h_s$.
(4) For every $i = s+1, \dots, n$ decompose $h_i = h_i^\dagger + \mathrm{Norm}(h_i)$ using $M = Q^\dagger \oplus \mathrm{Span}(\mathrm{NonTip}(Q^\dagger))$, i.e. using $h_1, \dots, h_s$ reduce h_i. After finite steps of reductions, we get $\mathrm{Norm}(h_i)$.
(5) Put $H = \{h_1, \dots, h_s, \mathrm{Norm}(h_{s+1}), \dots, \mathrm{Norm}(h_n)\}$.

Here, a reduction using $h_1, \dots, h_s$ means the following. If some element $m = \sum \alpha_i m_i$, $\alpha_i \in \mathbb{C}$, $m_i \in \mathcal{M}$ and, for some i and some $k \in \{1, \dots, s\}$, $\mathrm{Tip}(h_k)$ left divides m_i (i.e. there is $b \in \mathcal{B}$, $\alpha \in \mathbb{C}$ such that $m_i = \alpha\,\mathrm{Tip}(h_k) b$) then we can make a reduction. This means that we replace m_i in the decomposition of m by the element $m_i - \alpha h_k b$. The total reduction of m by $h_1, \dots, h_s$ is a sequence of elements of $\mathcal{M}$ such that each subsequent element is a reduction using $h_1, \dots, h_s$ of the preceding one and such that the last element cannot be further reduced. The last element will be $\mathrm{Norm}(m)$. It was proved in [Gre00, Proposition 4.2] that any finitely generated submodule $\mathcal{N}$ of a projective module over a path algebra has a right Gröbner basis which can be computed by the above algorithm.

An algebra A with an ordered multiplicative basis $(\mathcal{B}, >)$ is a right projective module over A with respect to the multiplication from the right. Putting $\mathcal{M} = \mathcal{B}$ we get a coherent module which has an admissible order $\succ$ equal to $>$. We will denote this module by A_r. In particular, every right ideal J is a submodule of A_r and, hence, has a right Gröbner basis.

Theorem 10. *Let $L_k \in V_{2k}^*$ be a flat functional on a path $*$-algebra $A = \mathcal{A}_\Gamma$. Let J be the right ideal generated by $\ker B_{L_k}$ in A. Then, there is a right Gröbner basis $\mathcal{G}$ such that $\mathcal{G} \subseteq \ker B_{L_k}$.*

Proof. Take any generating set $\{h_1, \dots, h_n\}$ of $\ker B_{L_k}$. If $\mathrm{Tip}(h_i)$ left divides $\mathrm{Tip}(h_j)$, i.e. $\mathrm{Tip}(h_j) = \mathrm{Tip}(h_i)b$, where $b \in \mathcal{B}$, then $h_j - h_i b \in \ker B_{L_k}$ since $h_i b \in \ker B_{L_k}$ by Lemma 9. This proves that step (4) of the algorithm of computing right Gröbner basis transforms elements of $\ker B_{L_k}$ into elements of $\ker B_{L_k}$. All other steps of the algorithm, clearly, produce no new elements. Thus, the obtained right Gröbner bases will be contained in $\ker B_{L_k}$. $\square$

Theorem 11. *Any flat truncated functional L_k on a path $*$-algebras $A = \mathcal{A}_\Gamma$ has an extension to a linear functional on A such that*

$$\operatorname{rank} B_{L_k} = \operatorname{rank} B_L. \tag{11}$$

An extension which satisfies (11) *is unique and positive semidefinite if L_k is such.*

Proof. Let J be the right ideal generated by $\ker B_{L_k}$ in A. Let us denote by $\widetilde{B}_{L_k}$ the induced sesquilinear form on $V_k/\ker B_{L_k}$. The natural map $\phi : V_k/\ker B_{L_k} \to A/J$ is linear. The map ϕ is onto since, modulo $\ker B_{L_k}$, each element of V_k is equivalent to an element of V_{k-1} and hence, by induction, each element of A is equivalent to some element from V_{k-1} modulo the right ideal J. The same inductive argument shows that, for every $b \in S_{2k}$, there are $v \in V_{k-1}$, $g_t \in \ker B_{L_k}$ and $b_t \in S_k$ $(t = 1, \dots, l)$ such that

$$b = v + g_1 b_1 + \cdots + g_l b_l. \tag{12}$$

Hence,

$$L_k(b) = L_k(v) + B_{L_k}\big(g_1, b_1^*\big) + \cdots + B_{L_k}\big(g_l, b_l^*\big) = L_k(v). \tag{13}$$

By Theorem 10 we can find a Gröbner basis $\mathcal{G}$ for the right ideal J such that $\mathcal{G} \subseteq \ker B_{L_k}$. If $x \in V_k$ is such that $\phi(x) = 0$ (i.e. $x \in J$) then, since J is a right Gröbner basis, we have

$$x = \sum_{j=1}^{s} \alpha_j g_j b_j, \tag{14}$$

where $\alpha_j \in \mathbb{C}^*$, $g_j \in \mathcal{G}$ and $b_j \in \mathcal{B}$. Moreover, we can assume, without loss of generality, that $\mathrm{Tip}(g_j) b_j \neq 0$ and $g_j b_j \in V_k$. Indeed, by the definition of a right Gröbner basis, there is $g \in \mathcal{G}$ such that $\mathrm{Tip}(x) = \alpha\, \mathrm{Tip}(g) b$ for some $b \in \mathcal{B}$ and $\alpha \in \mathbb{C}^*$. Thus, $\mathrm{Tip}(g) b \neq 0$, the element $y = x - \alpha g b$ belongs to J and $\mathrm{Tip}(y)$ is strictly less than $\mathrm{Tip}(x)$. Hence, by induction, we get that x has required decomposition (14) with $\mathrm{Tip}(g_j) b_j \neq 0$ for all j. By Lemma 9 we get that $g_j b_j \in \ker B_{L_k}$ for all $j = 1, \dots, s$. Consequently, $x \in \ker B_{L_k}$. Thus, ϕ is injective. By (12) we also have

$$L_k(p) = L_k\big(\phi^{-1}(p + J)\big) \quad \text{for all } p \in V_{2k}. \tag{15}$$

Thus, putting $L(p) = L_k(\phi^{-1}(p + \mathcal{J}))$ for $p \in A$, we get an extension of L_k.

Let us show that $\ker B_L = J$. For every $p \in J$ and $q \in A$, we have $pq \in J$. Thus, $B_L(p, q) = 0$. This implies that $J \subseteq \ker B_L$. To prove the converse inclusion, consider $q \in \ker B_L$ and take $p \in V_k$ such that $p - q \in J$. Since $J \subseteq \ker B_L$, we have $p \in \ker B_L \cap V_k$. Hence, $p \in \ker B_{L_k} \subseteq J$, and we conclude that $\ker B_L = J$.

For any $p_1, p_2 \in A$ we can find $q_1, q_2 \in V_k$ such that the elements $d_j = p_j - q_j$ belong to V_k. Then, $L(d_1 p_2^*) = 0$ which implies that $L(p_1 p_2^*) = L(q_1 p_2^*)$. Analogously, $L(q_1 d_2^*) = 0$ which implies $L(q_1 p_2^*) = L(q_1 q_2^*)$. We conclude that $L(p_1 p_2^*) = L_k(q_1 q_2^*)$. From this follows that L is hermitian and positive definite if L_k is such.

Let us prove uniqueness. Let $L' \in A^*$ be a linear extension of L_k to A such that $\operatorname{rank} B_{L'} = \operatorname{rank} B_{L_k}$. The subspace $\ker B_{L'} \cap V_k = \{v \in V_k \mid L'(vp^*) = 0 \text{ for all } p \in A\}$ is, clearly, contained in $\ker B_{L'_k}$. The latter is equal to $\ker B_{L_k}$ since L is an extension of L_k. Thus, we have the following diagram

$$A/\ker B_{L'} \xleftarrow{\;i\;} V_k/\ker B_{L'} \cap V_k \xrightarrow{\;\pi\;} V_k/\ker B_{L_k} \cong A/\ker B_L. \tag{16}$$

The dimensions in the above diagram are decreasing from left to right, and we get that $\ker B_{L'} \cap V_k = \ker B_{L_k}$ and i is surjective. From this follows that for any $p \in A$, there exists $q \in V_k$ such that $p - q \in \ker B_{L'}$. Hence, $L'(p) = L'(q) = L_k(q)$. $\square$

Lemma 12. *Let $L_k \in V_{2k}^*$ be hermitian. Decompose B_{L_k} into the block-matrix*

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

with respect to the decomposition $V_k = V_{k-1} \oplus \operatorname{Span}(S_k \setminus S_{k-1})$. Then, L_k is flat if and only if

$$\operatorname{Ran}(C) \subseteq \operatorname{Ran}(A) \quad \text{and} \quad B = C^*(A|_{\operatorname{Ran}(A)})^{-1}C.$$

If $B_{L_{k-1}}$ is positive semidefinite then so is B_{L_k}.

Proof. Assume first that L_k is flat. By (10) for every $b \in S_k \setminus S_{k-1}$ there is $v \in V_{k-1}$ such that $b - v \in \ker B_{L_k}$ and, hence,

$$Av = Cb,$$
$$C^*v = Bb.$$

Since the set of all b generates $\operatorname{Span}(S_k \setminus S_{k-1})$, which is a domain of C, the first equation above means that $\operatorname{Ran}(C) \subseteq \operatorname{Ran}(A)$. If $\operatorname{Ran}(C) \subseteq \operatorname{Ran}(A)$ then the operator $(A|_{\operatorname{Ran}(A)})^{-1}C$ is well defined and $v = (A|_{\operatorname{Ran}(A)})^{-1}Cb$. Hence, $Bb = C^*(A|_{\operatorname{Ran}(A)})^{-1}Cb$. Thus, $B = C^*(A|_{\operatorname{Ran}(A)})^{-1}C$.

Assume now that $\operatorname{Ran}(C) \subseteq \operatorname{Ran}(A)$ and $B = C^*(A|_{\operatorname{Ran}(A)})^{-1}C$. For every $b \in S_k \setminus S_{k-1}$ we can put $v = (A|_{\operatorname{Ran}(A)})^{-1}Cb$. It follows that $V_k = \ker B_{L_k} + V_{k-1}$. We need to show that $\ker B_{L_{k-1}} \subseteq \ker B_{L_k}$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product on the vector space of the algebra A such that $\mathcal{B}$ is an orthonormal basis. In particular $A_{b_1,b_2} = \langle b_1, Ab_2 \rangle$. If $v \in \ker B_{L_{k-1}}$ then $Av = 0$. Hence, $\langle v, Av' \rangle = 0$ for all $v' \in V_{k-1}$. Thus, v is orthogonal to $\operatorname{Ran}(A)$ and, consequently, to $\operatorname{Ran}(C)$. Hence $\langle v, Cw \rangle = 0$ for all $w \in \operatorname{Span}(S_k \setminus S_{k-1})$. From this follows that $B_{L_k}(v, w) = \langle v, Cw \rangle = 0$ and $\ker B_{L_{k-1}} \subseteq \ker B_{L_k}$. Form this and the equality $V_k = \ker B_{L_k} + V_{k-1}$, we get $\dim \ker B_{L_k} = \dim \ker A + \dim \operatorname{Span}(S_k \setminus S_{k-1})$. Thus, the non-zero eigenvalues of B_{L_k} are exactly the non-zero eigenvalues of the matrix A. $\square$

For the class of free $*$-algebras some flat functionals can be obtained as extensions of tip–maximal functionals defined below. It is easy to construct some tip–maximal functionals which will be used in the examples.

Definition 13. Let us call a hermitian functional L_k *tip–maximal* if there is a generating set $p_1, \ldots, p_r$ for $\ker B_{L_k}$ such that the projections of p_k on $\operatorname{Span}(S_k \setminus S_{k-1})$ parallel to $\operatorname{Span}(S_{k-1})$ are linearly independent. In particular, L_k is tip–maximal if B_{L_k} is non-degenerate.

Lemma 14. *Every tip–maximal L_{k-1} on a free $*$-algebra can be extended to a flat L_k. Moreover, if L_{k-1} is positive semidefinite then L_k is such.*

Proof. We will construct the matrix of L_k in the block-matrix form

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

with respect to the decomposition $V_k = V_{k-1} \oplus \operatorname{Span}(S_k \setminus S_{k-1})$.

First, we need to define L_k on the words of length $2k-1$. Thus, we will define the corner C in the above matrix. Let us enumerate the words of length $k-2$ by $z_1, \dots, z_t$, and the words of length $k-1$ by $w_1, \dots, w_s$. By the definition of the tip–maximal functional, we can find generators of $\ker A = \ker B_{L_{k-1}}$ of the form $\sum_i \alpha_i^{(m)} w_i - \sum_j \beta_j^{(m)} z_j$ $(m = 1, \dots, r)$ such that elements $f_m = \sum_i \alpha_i^{(m)} w_i$ are linearly independent. Thus, for every word $v \in S_{k-1}$, we have

$$\sum_i \alpha_i^{(m)} L_{k-1}(w_i v) = \sum_j \beta_j^{(m)} L_{k-1}(z_j v). \tag{17}$$

Clearly, the set S_{2k-1} coincides with the set of words $w_i w$ where $|w| = k$ and $i = 1, \dots, s$. For each $w_i w$, we introduce a variable $x_{i,w}$, and for each $j = 1, \dots, t$, a constant $a_{j,w} = L_{k-1}(z_j w)$ ($z_j w \in S_{2k-2}$, hence, $L_{k-1}(z_j w)$ is defined). Consider the system of linear equations

$$\sum_i \alpha_i^{(m)} x_{i,w} = \sum_j \beta_j^{(m)} a_{j,w} \quad (m = 1, \dots, r). \tag{18}$$

Since the family $w_1 w, \dots, w_s w$ and the family $f_1, \dots, f_r$ are both linearly independent, the matrix $(\alpha_i^{(m)})_{i,m}$ has rank r. Thus, system (18) has a solution. Given a solution $x_{j,m}$, define

$$L_k(v) = \begin{cases} L_{k-1}(v) & \text{for } v \in S_{k-1}, \\ x_{i,w} & \text{for } v = w_i w \text{ with some } w \text{ such that } |w| = k. \end{cases}$$

Thus, we have defined L_k on the subspace V_{2k-1}. Hence, the operator $C : \operatorname{Span}(S_k \setminus S_{k-1}) \to V_{k-1}$ in the block-matrix form of L_k is defined by the matrix with (v, w)-entry equal to $L_k(vw^*)$, where $v \in V_{k-1}$ and $w \in S_k \setminus S_{k-1}$. By the definition of $x_{i,w}$, we immediately have that

$$\sum_i \alpha_i^{(m)} L_k(w_i w) = \sum_j \beta_j^{(m)} L_k(z_j w) \tag{19}$$

for $w \in S_k \setminus S_{k-1}$. Since L_k is an extension of L_{k-1}, condition (19) is satisfied also for all $w \in S_{k-1}$. Since, for every generator $\sum_i \alpha_i^{(m)} w_i - \sum_j \beta_j^{(m)} z_j$ of $\ker B_{L_{k-1}}$, Eq. (17) implies (19), we get that the subspace generated by the vectors

$$\bigl(L_{k-1}(z_1 v), \dots, L_{k-1}(z_s v), L_{k-1}(w_1 v), \dots, L_{k-1}(w_r v)\bigr),$$

where $v \in V_{k-1}$, contains the subspace generated by the vectors

$$\bigl(L_k(z_1 u), \dots, L_k(z_s u), L_k(w_1 u), \dots, L_k(w_r u)\bigr),$$

where $u \in \operatorname{Span}(S_k \setminus S_{k-1})$. Thus $\operatorname{Ran}(C) \subseteq \operatorname{Ran}(A)$.

Since the words uv^* with $|u| = |v| = k$ are distinct for distinct pairs (u, v), they constitute a linearly independent family (which is, in fact, equal to $S_{2k} \setminus S_{2k-1}$). Thus, we can define $L_k(uv^*) = B_{u,v}$, where $B = C^*(A|_{\operatorname{Ran}(A)})^{-1} C$. By Lemma 12, L_k is flat and is positive semidefinite if L_{k-1} is such. $\square$

As a corollary, we have the following

Theorem 15. *Every positive semidefinite truncated tip–maximal functional on a free $*$-algebra $\mathcal{F}$ has a positive semidefinite extension to $\mathcal{F}$.*

Proof. Any positive semidefinite truncated tip–maximal functional L_k can be extended to a flat positive semidefinite L_{k+1} by Lemma 14. Then, L_{k+1} can be extended again to a positive semidefinite L on $\mathcal{F}$ by Theorem 11. $\square$

Example. Consider the $*$-algebra $A = \mathbb{C}\langle x, x^* \mid x^2 = 0,\ x^{*2} = 0\rangle$. Then $S_3 = \{x, x^*, xx^*, x^*x, xx^*x, x^*xx^*\}$. By a direct calculation we have that any hermitian L_3 with real coefficients has a form

$$\begin{pmatrix} a_1 & 0 & 0 & a_3 & a_5 & 0 \\ 0 & a_2 & a_4 & 0 & 0 & a_6 \\ 0 & a_4 & a_5 & 0 & 0 & a_7 \\ a_3 & 0 & 0 & a_6 & a_8 & 0 \\ a_5 & 0 & 0 & a_8 & a_9 & 0 \\ 0 & a_6 & a_7 & 0 & 0 & a_{10} \end{pmatrix}.$$

By Lemma 12, L_3 is flat if and only if the following two conditions hold:

$$a_9 = \frac{a_6^2 a_5 - 2a_3 a_5 a_8 + a_1 a_8^2}{a_1 a_6 - a_3^2}, \tag{20}$$

$$a_{10} = \frac{a_5 a_8^2 - 2a_4 a_6 a_7 + a_2 a_7^2}{a_2 a_5 - a_4^2}. \tag{21}$$

One can check that L_2 is positive definite iff $a_2 > 0$, $a_6 > 0$, $a_2 a_5 > a_4^2$ and $a_1 a_6 > a_3^2$. Thus, any positive L_2 can be extended to a positive semidefinite flat L_3.

Example. Consider the free $*$-algebra $\mathcal{F} = \mathbb{C}\langle x, x^*\rangle$. Denote by $\mathcal{B}_+$ the set of words of the form ww^*, where w is an arbitrary word in generators x and x^*. It was shown in [Pop10] that there are faithful positive functionals on $\mathcal{F}$ (as well as on any Lance–Tapper $*$-algebra) of the form $a \mapsto F(\Pi(a))$, where $\Pi : \mathcal{F} \to \mathcal{F}$ is the unique linear extension of the map $\Pi : \mathcal{B} \to \mathcal{B}$ defined via the rule

$$\Pi(w) = \begin{cases} w & \text{if } w \in \mathcal{B}_+, \\ 0 & \text{if } w \notin \mathcal{B}_+; \end{cases}$$

and $F : \mathbb{C}\mathcal{B}_+ \to \mathbb{C}$ is a linear functional defined on the subspace generated by $\mathcal{B}_+$.

Assume that $L_4 \in V_8^*$ is such that

$$L_4|_{V_7} \text{ is of the form } F(\Pi|_{V_7}) \text{ for some functional } F \in V_7^*, \tag{22}$$

and the matrix of B_{L_4} has a block matrix decomposition

$$\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$$

such that the matrix A is of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$$

with $A_{11} = I_7$ (the identity (7×7)-matrix), $A_{22} = \operatorname{diag}(2, 3, 1, 1, 2, 3, 0)$ and

$$A_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By routine calculation, one can check that the form on $V_3 \times V_3$ defined by the matrix A comes from the functional of the form $F(\Pi|_{V_6})$, and is positive semidefinite. Condition (22) ensures that the matrix C is completely determined by A.

In the decomposition $V_3 = A(V_3) \oplus \ker A$, the subspace $A(V_3)$ is

$$\operatorname{Span}\{x, x^*, x^2, xx^*, x^*x, x^{*2}, x^3, x^2x^*, xx^*x, xx^{*2}, x^*x^2, x^*xx^*, x^{*2}x\}$$

and $\ker A = \operatorname{Span}\{x^{*3}\}$. The matrix A is chosen to satisfy three conditions: it is positive semidefinite, tip–maximal and annulates $(x^*)^3$. The matrix B and the functional L_4 is completely determined by Lemma 12.

The following set H

$$x^{*3}, x^4, x^3x^* - x^2, x^2x^*x - 2x^2, x^2x^{*2}, xx^*x^2, (xx^*)^2 - 3xx^*, xx^{*2}x - xx^*, xx^{*3},$$
$$x^*x^3, xx^{*3}, x^*x^3, x^*x^2x^* - x^*x, (x^*x)^2 - 2x^*x, x^*xx^{*2}, x^{*2}x^2, x^{*2}xx^* - 3x^{*2}, x^{*3}x, x^{*4}$$

is a linear basis of the space $\ker B_{L_4}$.

A right Gröbner basis $\mathcal{G}$ of the right ideal J generated by $\ker B_{L_4}$ is the following set

$$x^{*3}, x^4, x^3x^* - x^2, x^2x^*x - 2x^2, x^2x^{*2}, xx^*x^2, (xx^*)^2 - 3xx^*, xx^{*2}x - xx^*, xx^{*3},$$
$$x^*x^3, x^*x^2x^* - x^*x, (x^*x)^2 - 2x^*x, x^*xx^{*2}, x^{*2}x^2, x^{*2}xx^* - 3x^{*2}.$$

The only reductions we have made to compute the above Gröbner basis, starting from H, are the reductions of $x^{*3}x$ and x^{*4} to zero.

The positive definite form $\widetilde{B}_{L_4}$ (in the notations introduced in the proof of Theorem 11) is the form with a matrix $\widetilde{A}$ in the basis consisting of cosets of $\mathcal{F}/J$ with the representatives

$$x, x^*, x^2, xx^*, x^*x, x^{*2}, x^3, x^2x^*, xx^*x, xx^{*2}, x^*x^2, x^*xx^*, x^{*2}x.$$

The matrix $\widetilde{A}$ is obtained from the matrix A_{11} by deleting the first row and first column. The operators R_x and R_{x^*} of multiplications from the right by x and x^*, correspondingly, define the mutually adjoint operators $\widetilde{R}_x$ and $\widetilde{R}_{x^*}$ in the Hilbert space of right cosets $\mathcal{F}/J$ with the inner product given by the matrix $\widetilde{A}$. Thus, we get the 13-dimensional representation π of the free $*$-algebras $\mathcal{F}$. It can be checked that $\ker \pi \cap V_4$ is linearly generated by the following elements:

$$x^3, -5x^{*2} + 2x^*xx^{*2} + x^{*2}xx^*, x^*x^2x^* - xx^{*2}x.$$

In particular, π defines a representation of the Lance–Tapper $*$-algebra

$$\mathbb{C}\langle x, x^* \mid x^3 = 0,\ x^{*3} = 0\rangle.$$

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